

On strong approximation by logarithmic means of Fourier series

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Dedicated to Professor K. Tandori on his 60th birthday

Introduction. Let $L_{2\pi}^p$ ($1 < p < \infty$) be the class of all real-valued functions f , 2π -periodic, Lebesgue-integrable with p -th power over $\langle -\pi, \pi \rangle$.

Consider the Fourier series

$$S[f] = \frac{a_0(f)}{2} + \sum_{v=1}^{\infty} (a_v(f) \cos vx + b_v(f) \sin vx),$$

and denote by $S_k(x; f)$ and $\sigma_k^\alpha(x; f)$ the partial sums and (C, α) -means of $S[f]$, respectively, thus, e.g.,

$$\sigma_k^\alpha(x; f) = \frac{1}{A_k^\alpha} \sum_{v=0}^k A_{k-v}^{\alpha-1} S_v(x; f) \quad (\alpha > -1, k = 0, 1, 2, \dots),$$

where $A_k^\alpha = \binom{k+\alpha}{k}$.

DEOKINANDAN [1] proved the following theorem: If $f \in L_{2\pi}^2$, and for a fixed δ , the condition

$$\int_t^\delta \frac{|f(x+u) + f(x-u) - 2s|^2}{t} dt = o\left(\log \frac{1}{t}\right) \quad \text{as } t \rightarrow 0+$$

holds, then

$$(*) \quad \sum_{k=1}^{\infty} \frac{r^k}{k} |S_k(x; f) - s|^2 = o\left(\log \frac{1}{1-r}\right) \quad \text{as } r \rightarrow 1-.$$

In this paper we shall generalize and extend this theorem by taking the functions $f \in L_{2\pi}^p$ ($1 < p < \infty$) and replacing the partial sums $S_k(x; f)$ in (*) by (C, α) -means with negative α . More precisely, we shall estimate the quantity

$$H_n^{\log}(x; \alpha, f)_q = \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} |\sigma_k^\alpha(x; f) - f(x)|^q \right\}^{1/q};$$

as a measure of this deviation we introduce the function

$$w_x^{\log}(\delta; f)_{p,q} = \sup_{0 < h \leq \delta} \left\{ \frac{1}{\log^{p/q}(h^{-1}+1)} \int_h^\pi \frac{1}{t} |\varphi_x(t)|^p dt \right\}^{1/p},$$

where $\varphi_x(t) = \varphi_x(t; f) = f(x+t) + f(x-t) - 2f(x)$.

We shall show that our results cannot be improved for some classes of functions, too.

An analogous problem in the case of Riesz means was raised by LEINDLER [4] and solved by TOTIK [5].

By $C_j(\cdot)$ ($j=1, 2, 3, \dots$) we signify positive constants depending on the indicated parameters, only.

Statements of results. First, we give the estimate for $\alpha=0$.

Theorem 1. *If $f \in L_{2\pi}^p$ ($1 < p \leq q < \infty$), then*

$$H_n^{\log}(x; 0, f)_{\bar{q}} \leq C_1(p, q) w_x^{\log}\left(\frac{1}{n}; f\right)_{p,q}$$

for $n=1, 2, 3, \dots$ and $\bar{q} \in (0, \pi)$.

An interesting case is if $p=q=\bar{q}$. Then this result cannot be improved in the following sense.

Let $L^p(\Omega)$ be the subclass of $L_{2\pi}^p$, generated by a nonnegative and nondecreasing function Ω defined on $\langle 0, \pi \rangle$, with $\Omega(0+)=0$ and $\Omega(t)>0$ for any $t < \pi$, consisting of all functions $g \in L_{2\pi}^p$ such that

$$M_g = M_g(x) = \sup_{0 < \delta \leq \pi} \left\{ w_x^{\log}(\delta; g)_{p,p} \left(\frac{1}{\log(\delta^{-1}+1)} \int_\delta^\pi \frac{\Omega^p(t)}{t} dt \right)^{-1/p} \right\} < \infty,$$

and let

$$L_M^p(\Omega) = \{g: M_g < M, g \in L^p(\Omega), M = \text{constant} > 0\}.$$

Theorem 2. *If $t^{-1}\Omega(t)$ is a nonincreasing function of t , then there exists an absolute constant C_2 ($< C_1(p, p)$) such that*

$$MC_2 \leq \sup_{f \in L_M^p(\Omega)} \left\{ H_n^{\log}(x; 0, f)_p \left(\frac{1}{\log(n+1)} \int_{1/n}^\pi \frac{\Omega^p(t)}{t} dt \right)^{-1/p} \right\} \leq MC_1(p, p)$$

for $n=1, 2, 3, \dots$ and $1 < p < \infty$.

For $\alpha \in (-1/2, 0)$, we have the following result.

Theorem 3. Let $f \in L_{2n}^p$, $1/(1+\alpha) \leq p \leq -1/\alpha$ and $-1/2 \leq \alpha < 0$; then

$$H_n^{\log}(x; \alpha, f)_{\bar{q}} \leq C_3(p, q, \alpha) \sup_{1/(1+\alpha) \leq \bar{p} \leq p} \left\{ w_x^{\log} \left(\frac{1}{n}; f \right)_{\bar{p}, q} \right\}$$

for $n=1, 2, 3, \dots$ and $p \leq q \leq -1/\alpha$, $\bar{q} \in (0, q)$.

For $\alpha = -1/2$ Theorem 3 gives the following improvement of Deokinandan's result.

Corollary. If $f \in L_{2n}^2$, then

$$H_n^{\log}(x; -1/2, f)_{\bar{q}} \leq C_3(2, 2, -1/2) w_x^{\log} \left(\frac{1}{n}; f \right)_{2, 2}$$

for $n=1, 2, 3, \dots$ and $\bar{q} \in (0, 2)$.

In the special case $p=q=\bar{q}=1/(1+\alpha)$ ($-1/2 \leq \alpha < 0$) there holds a theorem of the same type as Theorem 2:

Theorem 4. If $t^{-1}\Omega(t)$ is a nonincreasing function of t , then we can define the constant $C_4(\alpha)$ less than $C_3(p, q, \alpha)$ so that the following inequalities

$$MC_4(\alpha) \leq \sup_{f \in L_{M(\Omega)}^p} \left\{ H_n^{\log}(x; \alpha, f)_p \left(\frac{1}{\log(n+1)} \int_{1/n}^{\pi} \frac{\Omega^p(t)}{t} dt \right)^{-1/p} \right\} \leq MC_3(p, p, \alpha)$$

are true, whenever $-1/2 \leq \alpha < 0$, $p=1/(1+\alpha)$ and $n=1, 2, 3, \dots$

In connection with the above theorem we formulate a statement.

Remark. Using the method of TOTIK [6] we can prove analogously that the estimate obtained in Theorem 3, for $p=q=\bar{q}=1/(1+\alpha)$, is the best possible in the sense considered in [6], because the logarithmic method satisfies the desired condition.

Auxiliary results. Let us start with the following inequality of Hardy—Littlewood—Pólya (H—L—P).

Theorem A. If $q > 1$ and $-1/2 \leq \alpha < 0$ such that $-1 \leq \alpha q$, then

$$\left\{ \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{n^{-\alpha-1/q}}{(m+n)^{1-\alpha}} |d_m| \right)^q \right\}^{1/q} \leq \frac{\pi}{\sin(-\alpha\pi)} \left\{ \sum_{m=1}^{\infty} \frac{|d_m|^q}{m} \right\}^{1/q}$$

for any sequence $\{d_n\}$ of real numbers.

This inequality can be deduced from the general inequality of H—L—P ([3] Theorem 318, p. 227) but, in our special case, it is easier to give the direct proof.

Namely,

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{n^{-\alpha-1/q}}{(m+n)^{1-\alpha}} |d_m| \right)^q &\leq \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{n^{-\alpha-1/q}}{(m+n)^{m^{-\alpha}}} |d_m| \right)^q = \\ &= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(\frac{1}{m+n} \right)^{1/q} \frac{|d_m|}{m^{1/q}} \left(\frac{m}{n} \right)^{\alpha-\beta+1/q} \left(\frac{1}{m+n} \right)^{1/q'} \left(\frac{n}{m} \right)^{-\beta} \right)^q, \end{aligned}$$

where $\alpha < \beta < 0$ and $1/q + 1/q' = 1$. Hence, by Hölder inequality, the left-hand side of our inequality does not exceed

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} \frac{|d_m|^q}{m(m+n)} \left(\frac{m}{n} \right)^{1+(\alpha-\beta)q} \left(\sum_{k=1}^{\infty} \frac{1}{k+n} \left(\frac{n}{k} \right)^{-\beta q'} \right)^{q/q'} \right\} = \\ &= \sum_{m=1}^{\infty} \left\{ \frac{|d_m|^q}{m} \sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n} \right)^{1+(\alpha-\beta)q} \left(\sum_{k=1}^{\infty} \frac{1}{k+n} \left(\frac{n}{k} \right)^{-\beta q'} \right)^{q/q'} \right\} \leq \\ &\leq \sum_{m=1}^{\infty} \left\{ \frac{|d_m|^q}{m} \sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n} \right)^{1+(\alpha-\beta)q} \left(\int_0^{\infty} \frac{x^{\beta q'}}{1+x} dx \right)^{q/q'} \right\} \leq \\ &\leq \sum_{m=1}^{\infty} \frac{|d_m|^q}{m} \left(\int_0^{\infty} \frac{y^{-(1+\alpha q - \beta q)}}{1+y} dy \right) \left(\int_0^{\infty} \frac{x^{\beta q'}}{1+x} dx \right)^{q/q'} = \\ &= \frac{\pi^{1+q/q'}}{\sin(\pi(1+\alpha q - \beta q))(\sin(-\pi \beta q'))^{q/q'}} \sum_{m=1}^{\infty} \frac{|d_m|^q}{m}, \end{aligned}$$

if $0 < 1 + \alpha q - \beta q < 1$ and $0 < -\beta q' < 1$ (cf. [3], (9.2.1-2) p. 228).

Taking $\beta = \alpha/q'$, we obtain that $1 + \alpha q - \beta q = 1 + \alpha$ and $\sin(\pi(1+\alpha)) = \sin(-\pi\alpha)$; whence we have our inequality with the desired constant.

We require the following inequality, too.

Theorem B ([3], Theorem 346, p. 255). *If $\beta > 1$ and $p > 1$, then*

$$\sum_{n=1}^{\infty} n^{-\beta} D_n^p \leq C_5(\beta, p) \sum_{n=1}^{\infty} n^{-\beta} (nd_n)^p$$

for any positive sequence $\{d_n\}$ and $D_n = d_1 + d_2 + \dots + d_n$.

The following two theorems of Hardy and Littlewood will be needed, too.

Theorem C ([7] Theorem 5.20, Ch. XII). *If $h \in L_{2\pi}^r$ and $r \leq s \leq r'$ ($r' \geq 2$), then*

$$\left\{ \frac{|a_0(h)|^s}{2} + \sum_{n=1}^{\infty} (n+1)^{-\lambda s} (|a_n(h)|^s + |b_n(h)|^s) \right\}^{1/s} \leq C_6(r) \left\{ \int_{-\pi}^{\pi} |h(t)|^r dt \right\}^{1/r},$$

where $\lambda = 1/s + 1/r - 1$ and $1/r + 1/r' = 1$.

Theorem D ([2] Theorem 10, p. 369). *If $1 < p \leq q$ and $|t|^{-1}|\varphi_x(t)|^p \in L_{2\pi}^1$, then*

$$\left\{ \sum_{k=1}^{\infty} \frac{1}{k} |S_k(x; f) - f(x)|^q \right\}^{1/q} \leq C_7(p, q) \left\{ \int_0^{\pi} \frac{|\varphi_x(t)|^p}{t} dt \right\}^{1/p}.$$

Proof of Theorem 1. The relation

$$S_k(x; f) - f(x) = \frac{2}{\pi} \int_0^{\pi} \varphi_x(t) D_k(t) dt,$$

where $D_k(t)$ denote the Dirichlet's kernel, gives

$$\begin{aligned} & \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} |S_k(x; f) - f(x)|^q \right\}^{1/q} \leq \\ & \leq \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{2}{\pi} \int_0^{1/n} \varphi_x(t) D_k(t) dt \right|^q \right\}^{1/q} + \\ & + \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{2}{\pi} \int_{1/n}^{\pi} \varphi_x(t) D_k(t) dt \right|^q \right\}^{1/q} = I_1 + I_2. \end{aligned}$$

Since

$$|D_k(t)| \leq k + 1/2 < 2k,$$

we have

$$\begin{aligned} I_1 & \leq 2 \frac{2}{\pi} \int_0^{1/n} |\varphi_x(t)| dt \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n k^{q-1} \right\}^{1/q} \leq \\ & \leq \frac{2n}{\log^{1/q}(n+1)} \frac{2}{\pi} \int_0^{1/n} |\varphi_x(t)| dt \leq \frac{4}{\pi} \left\{ \frac{n}{\log^{p/q}(n+1)} \int_0^{1/n} |\varphi_x(t)|^p dt \right\}^{1/p}. \end{aligned}$$

We observe

$$\left\{ \frac{n}{2} \int_0^{2/n} |\varphi_x(t)|^p dt \right\}^{1/p} \leq \left\{ \frac{1}{2} n \int_0^{1/n} |\varphi_x(t)|^p dt \right\}^{1/p} + \left\{ \int_{1/n}^{2/n} \frac{|\varphi_x(t)|^p}{t} dt \right\}^{1/p},$$

and, further

$$\begin{aligned} & \sup_{0 < u \leq 1/n} \left\{ \frac{1}{2u \log^{p/q} \left(\frac{1}{2u} + 1 \right)} \int_0^{2u} |\varphi_x(t)|^p dt \right\}^{1/p} \leq \\ & \leq 2^{-1/p} \sup_{0 < u \leq 1/n} \left\{ \frac{1}{u \log^{p/q} \left(\frac{1}{2u} + 1 \right)} \int_0^u |\varphi_x(t)|^p dt \right\}^{1/p} + \\ & + \sup_{0 < u \leq 1/n} \left\{ \frac{1}{\log^{p/q} \left(\frac{1}{2u} + 1 \right)} \int_u^{\pi} \frac{|\varphi_x(t)|^p}{t} dt \right\}^{1/p}. \end{aligned}$$

Hence, because

$$\log\left(\frac{1}{2u}+1\right) \equiv \left(\frac{\log 3}{\log 2}-1\right) \log\left(\frac{1}{u}+1\right) \quad \text{if } 0 < u \leq \frac{1}{n} \quad (n = 1, 2, 3, \dots),$$

we obtain

$$\begin{aligned} & \sup_{0 < u \leq 1/n} \left\{ \frac{1}{u \log^{p/q}\left(\frac{1}{2u}+1\right)} \int_0^u |\varphi_x(t)|^p dt \right\}^{1/p} \equiv \\ & \equiv \sup_{0 < v \leq 2/n} \left\{ \frac{1}{v \log^{p/q}\left(\frac{1}{v}+1\right)} \int_0^v |\varphi_x(t)|^p dt \right\}^{1/p} = \\ & = \sup_{0 < 2u \leq 2/n} \left\{ \frac{1}{2u \log^{p/q}\left(\frac{1}{2u}+1\right)} \int_0^{2u} |\varphi_x(t)|^p dt \right\}^{1/p} \equiv \\ & \equiv 2^{-1/p} \left(\frac{\log 3}{\log 2}-1\right)^{-1/q} \sup_{0 < u \leq 1/n} \left\{ \frac{1}{u \log^{p/q}\left(\frac{1}{u}+1\right)} \int_0^u |\varphi_x(t)|^p dt \right\}^{1/p} + \\ & + \left(\frac{\log 3}{\log 2}-1\right)^{-1/q} \sup_{0 < u \leq 1/n} \left\{ \frac{1}{\log^{p/q}\left(\frac{1}{u}+1\right)} \int_u^\pi \frac{|\varphi_x(t)|^p}{t} dt \right\}^{1/p}. \end{aligned}$$

Therefore

$$I_1 \equiv \frac{4}{\pi} \left\{ \left(\frac{\log 3}{\log 2}-1\right)^{1/q} - 2^{-1/p} \right\}^{-1} w_x^{\log}\left(\frac{1}{n}; f\right)_{p,q} = C_8(p, q) w_x^{\log}\left(\frac{1}{n}; f\right)_{p,q}.$$

To estimate of the second integral we apply Theorem D. Then

$$\begin{aligned} I_2 &= \frac{2}{\pi} \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \int_{1/n}^\pi \varphi_x(t) D_k(t) dt \right|^q \right\}^{1/q} = \\ &= \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} |S_k(0; f^*) - f^*(0)|^q \right\}^{1/q} \equiv \\ &\equiv C_7(p, q) \left\{ \frac{1}{\log^{p/q}(n+1)} \int_0^\pi \frac{|\varphi_0(t; f^*)|^p}{t} dt \right\}^{1/p} = \\ &= C_7(p, q) \left\{ \frac{1}{\log^{p/q}(n+1)} \int_{1/n}^\pi \frac{|\varphi_x(t)|^p}{t} dt \right\}^{1/p}, \end{aligned}$$

where $f^*(t) = f(x)$ for $t \in (-1/n, 1/n)$ and $f^*(t) = f(x+t)$ otherwise.

Summing the above estimates we have the desired result with the constant $C_3(p, q) = C_8(p, q) + C_7(p, q)$.

Proof of Theorem 2. Since, for $f \in L_M^p(\Omega)$,

$$w_x^{\log} \left(\frac{1}{n}; f \right)_{p,p} \leq M \left\{ \frac{1}{\log(n+1)} \int_{1/n}^{\pi} \frac{\Omega^p(t)}{t} dt \right\}^{1/p},$$

we have the first inequality, immediately.

To prove the second one, let us consider the function

$$f_x(t) = \frac{M}{4(8\pi+1)} \sum_{k=1}^{\infty} \left\{ \Omega\left(\frac{1}{k}\right) - \Omega\left(\frac{1}{k+1}\right) \right\} \cos k(t-x).$$

By our assumption, $u^{-1}\Omega(u) \leq (\lambda u)^{-1}\Omega(\lambda u)$ if $\lambda < 1$. Arguing as in [6], we see that

$$|\varphi_x(t; f_x)| \leq M\Omega(t),$$

and

$$w_x^{\log} \left(\frac{1}{n}; f_x \right)_{p,q} \leq M \left\{ \frac{1}{\log^{p/q}(n+1)} \int_{1/n}^{\pi} \frac{\Omega^p(t)}{t} dt \right\}^{1/p},$$

which gives $f_x \in L_M^p(\Omega)$.

Hence, in view of

$$S_k(x; f_x) - f_x(x) = \frac{M}{4(8\pi+1)} \Omega\left(\frac{\pi}{k+1}\right),$$

we obtain

$$\begin{aligned} \sup_{f \in L_M^p(\Omega)} H_n^{\log}(x; 0, f)_p &\cong H_n^{\log}(x; 0, f_x)_p \cong \frac{M}{4(8\pi+1)} \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{\Omega^p\left(\frac{\pi}{k+1}\right)}{k} \right\}^{1/p} \\ &\cong \frac{M}{4(8\pi+1)} \left\{ \frac{1}{\log(n+1)} \frac{1}{8} \int_1^{8n+1} \frac{\Omega^p\left(\frac{\pi}{u+1}\right)}{u+1} du \right\}^{1/p} \cong C_2 M \left\{ \frac{1}{\log(n+1)} \int_{1/n}^{\pi} \frac{\Omega^p(t)}{t} dt \right\}^{1/p}. \end{aligned}$$

Thus Theorem 2 is established.

Proof of Theorem 3. Since

$$\sigma_n^\alpha(x; f) - f(x) = \frac{2}{\pi} \int_0^\pi \varphi_x(t) K_n^\alpha(t) dt,$$

where $K_n^\alpha(t)$ denotes the (C, α) -kernel, and

$$|K_n^\alpha(t)| \leq 2n,$$

the proof reduces to estimate the second term in the following expression

$$\begin{aligned} & \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} |\sigma_k^\alpha(x; f) - f(x)|^q \right\}^{1/q} \equiv \\ & \equiv \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{2}{\pi} \int_0^{1/n} \varphi_x(t) K_k^\alpha(t) dt \right|^q \right\}^{1/q} + \\ & + \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{2}{\pi} \int_{1/n}^\pi \varphi_x(t) K_k^\alpha(t) dt \right|^q \right\}^{1/q} = J_1 + J_2. \end{aligned}$$

Here

$$J_1 \equiv \frac{2}{\pi} \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| 2k \int_0^{1/n} \varphi_x(t) dt \right|^q \right\}^{1/q} \equiv C_8(p, q) w_x^{\log} \left(\frac{1}{n}; f \right)_{p, q},$$

by the same argument as before.

Using the following form of the kernel

$$\begin{aligned} K_k^\alpha(t) &= \frac{1}{A_k^\alpha} \frac{\sin \left\{ (k+1/2+\alpha/2)t - \frac{\pi\alpha}{2} \right\}}{\left(2 \sin \frac{t}{2} \right)^{1+\alpha}} - \frac{1}{2A_k^\alpha \sin \frac{t}{2}} \operatorname{Im} \left(e^{i(k+1/2)t} \sum_{v=k+1}^\infty A_v^{\alpha-1} e^{-ivt} \right) = \\ &= \frac{\sin \left\{ (k+1/2+\alpha/2)t - \frac{\pi\alpha}{2} \right\}}{A_k^\alpha \left(2 \sin \frac{t}{2} \right)^{1+\alpha}} - \frac{1}{2A_k^\alpha \sin \frac{t}{2}} \sum_{v=k+1}^\infty A_v^{\alpha-1} \sin \left(k - v + \frac{1}{2} \right) t = \\ &= \frac{\sin \left\{ (k+1/2+\alpha/2)t - \frac{\pi\alpha}{2} \right\}}{A_k^\alpha \left(2 \sin \frac{t}{2} \right)^{1+\alpha}} + \frac{1}{A_k^\alpha} \sum_{\mu=k}^\infty A_{\mu+1}^{\alpha-1} D_{\mu-k}(t), \end{aligned}$$

we obtain

$$\begin{aligned} J_2 &\equiv \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{2}{\pi} \int_{1/n}^\pi \frac{\sin \left((k+1/2+\alpha/2)t - \frac{\pi\alpha}{2} \right)}{A_k^\alpha \left(2 \sin \frac{t}{2} \right)^{1+\alpha}} \varphi_x(t) dt \right|^q \right\}^{1/q} + \\ &+ \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{2}{\pi} \int_{1/n}^\pi \frac{1}{A_k^\alpha} \sum_{\mu=k}^\infty A_{\mu+1}^{\alpha-1} D_{\mu-k}(t) \varphi_x(t) dt \right|^q \right\}^{1/q} \equiv \\ &\equiv \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{2/\pi}{A_k^\alpha} \int_{1/n}^\pi \frac{\varphi_x(t) \cos \left(\frac{1+\alpha}{2} t - \frac{\pi\alpha}{2} \right)}{\left(2 \sin \frac{t}{2} \right)^{1+\alpha}} \sin kt dt \right|^q \right\}^{1/q} + \end{aligned}$$

$$+ \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{2/\pi}{A_k^\alpha} \int_{1/n}^{\pi} \frac{\varphi_x(t) \sin\left(\frac{1+\alpha}{2}t - \frac{\pi\alpha}{2}\right)}{\left(2 \sin \frac{t}{2}\right)^{1+\alpha}} \cos kt \, dt \right| \right\}^{1/q} +$$

$$+ \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{1}{A_k^\alpha} \sum_{\mu=k}^{\infty} A_{\mu+1}^{\alpha-1} \{S_{\mu-k}(0; f^*) - f^*(0)\} \right| \right\}^{1/q} = J_{21} + J_{22} + J_{23}.$$

The terms J_{21}, J_{22}, J_{23} will be estimated separately. First we consider the sum J_{23} . In view of

$$C_9(\alpha) k^\alpha \leq A_k^\alpha \leq C_{10}(\alpha) k^\alpha \quad (\alpha \neq -1, -2, \dots, k = 1, 2, \dots)$$

([7] (1.15) Ch. III) we have

$$J_{23} \leq \frac{C_{10}(\alpha-1)}{C_9(\alpha)} \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n k^{-\alpha q-1} \left(\sum_{v=0}^{\infty} (v+k+1)^{\alpha-1} |S_v(0; f^*) - f^*(0)|^q \right)^{1/q} \right\} \leq$$

$$= \frac{C_{10}(\alpha-1)}{C_9(\alpha)} \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \left(\sum_{v=1}^{\infty} \frac{k^{-\alpha-1/q}}{(v+k)^{1-\alpha}} |S_{v-1}(0; f^*) - f^*(0)|^q \right)^{1/q} \right\}.$$

Further, by Theorems A and D,

$$J_{23} \leq \frac{C_{10}(\alpha-1)}{C_9(\alpha)} \frac{\pi}{\sin(-\pi\alpha)} \left\{ \frac{1}{\log(n+1)} \sum_{v=1}^{\infty} \frac{1}{v} |S_{v-1}(0; f^*) - f^*(0)|^q \right\}^{1/q} \leq$$

$$\leq \frac{2^{1/q} \pi C_{10}(\alpha-1)}{C_9(\alpha) \sin(-\pi\alpha)} \left\{ \left(\frac{1}{\log(n+1)} \sum_{v=1}^{\infty} \frac{|S_v(0; f^*) - f^*(0)|^q}{v} \right)^{1/q} + \frac{\left| \frac{1}{2} a_0(f^*) - f^*(0) \right|}{\log^{1/q}(n+1)} \right\} \leq$$

$$\leq \frac{2^{1/q} C_{10}(\alpha-1)}{C_9(\alpha) \sin(-\pi\alpha)} \left\{ \left(\frac{1}{\log^{p/q}(n+1)} \int_{1/n}^{\pi} \frac{|\varphi_x(t)|^p}{t} \, dt \right)^{1/p} + \frac{1/\pi}{\log^{1/q}(n+1)} \int_{1/n}^{\pi} |\varphi_x(t)| \, dt \right\} \leq$$

$$\leq \frac{2^{1/q} C_{10}(\alpha-1) \pi (C_7(p, q) + 1)}{C_9(\alpha) \sin(-\pi\alpha)} \left\{ \frac{1}{\log^{p/q}(n+1)} \int_{1/n}^{\pi} \frac{|\varphi_x(t)|^p}{t} \, dt \right\}^{1/p}.$$

The estimates for J_{21} and J_{22} are similar, so we shall examine in detail the term J_{21} only. Then we can apply Theorem C with $\lambda = 1/q + \alpha \geq 0$, $s = q$, $r = 1/(1+\alpha)$, $r' = -1/\alpha$ and

$$h(t) = \begin{cases} \frac{\varphi_x(t) \sin\left(\frac{1+\alpha}{2}t - \frac{\alpha\pi}{2}\right)}{\left(2 \sin \frac{t}{2}\right)^{1+\alpha}} & \text{if } t \in \left(\frac{1}{n}, \pi\right), \\ 0 & \text{otherwise,} \end{cases}$$

and obtain

$$\begin{aligned}
 J_{21} &\cong \frac{2}{C_9(\alpha)} \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \left| k^{-\alpha-1/q} \frac{1}{\pi} \int_{-\pi}^{\pi} h(t) \sin kt \, dt \right|^q \right\}^{1/q} = \\
 &= \frac{2}{C_9(\alpha)} \log^{-1/q}(n+1) \left\{ \sum_{k=1}^n |k^{-\alpha-1/q} b_k(h)|^q \right\}^{1/q} \cong \\
 &\cong \frac{2C_6 \left(\frac{1}{1+\alpha} \right)}{C_9(\alpha)} \log^{-1/q}(n+1) \left\{ \int_{-\pi}^{\pi} |h(t)|^{1/(1+\alpha)} \, dt \right\}^{1+\alpha} \cong \\
 &\cong \frac{2C_6 \left(\frac{1}{1+\alpha} \right) (\pi/2)^{1+\alpha}}{C_9(\alpha)} \left\{ \frac{1}{\log^{1/q(1+\alpha)}(n+1)} \int_{1/n}^{\pi} \frac{|\varphi_x(t)|^{1/(1+\alpha)}}{t} \, dt \right\}^{1/\alpha}.
 \end{aligned}$$

Thus our proof is completed, Theorem 3 holds with the

$$\text{constant } C_3(p, q, \alpha) = C_8(p, q) + \frac{2^{1/q} C_{10}(\alpha-1) \pi (C_7(p, q) + 1)}{C_9(\alpha) \sin(-\pi\alpha)} + \frac{C_6 \left(\frac{1}{1+\alpha} \right) \pi^{1+\alpha}}{C_9(\alpha) 2^{-1+\alpha}}.$$

Proof of Theorem 4. The first estimate may be proved similarly as before.

To prove the second one let us consider the function f_x , too. By the identity

$$\sigma_k^{\alpha+1}(x; f) = \frac{1}{A_n^{\alpha+1}} \sum_{v=0}^k A_v^{\alpha} \sigma_v^{\alpha}(x; f) \quad (\text{cf. [7] Theorem 1.21, Ch. III}),$$

applying Theorem B, we get

$$\begin{aligned}
 &\sum_{k=1}^n \frac{1}{k} |\sigma_k^{\alpha}(x; f) - f(x)|^q \cong \\
 &\cong C_{10}^{-q}(\alpha) \sum_{k=1}^n k^{-(1+q(\alpha+1))} |k A_k^{\alpha}(\sigma_k^{\alpha}(x; f) - f(x))|^q \cong \\
 &\cong C_{10}^{-q}(\alpha) C_5^{-1}(1+q(\alpha+1), q) \sum_{k=1}^n k^{-(1+q(\alpha+1))} \left(\sum_{v=0}^k |A_v^{\alpha}(\sigma_v^{\alpha}(x; f) - f(x))| \right)^q \cong \\
 &\cong \frac{C_9^q(\alpha+1)}{C_{10}^q(\alpha) C_5(1+q(\alpha+1), q)} \sum_{k=1}^n \frac{1}{k} \left| \frac{1}{A_k^{\alpha+1}} \sum_{v=0}^k A_v^{\alpha}(\sigma_v^{\alpha}(x; f) - f(x)) \right|^q = \\
 &= C_{11}(\alpha, q) \sum_{k=1}^n \frac{1}{k} |\sigma_k^{1+\alpha}(x; f) - f(x)|^q.
 \end{aligned}$$

Hence, since

$$S_k(x; f_x) - f_x(x) = \frac{M}{4(8\pi+1)} \Omega\left(\frac{\pi}{k+1}\right),$$

we have

$$\begin{aligned}
 & \sup_{f \in L_M^p(\Omega)} H_n^{\log}(x; \alpha, f)_{1/(1+\alpha)} \cong H_n^{\log}(x; \alpha, f_x)_{1/(1+\alpha)} \cong \\
 & \cong C_{11} \left(\alpha, \frac{1}{1+\alpha} \right) \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} |\sigma_k^{\alpha+1}(x; f_x) - f_x(x)|^{1/(1+\alpha)} \right\}^{1+\alpha} = \\
 & = C_{11} \left(\alpha, \frac{1}{1+\alpha} \right) \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{1}{A_k^{\alpha+1}} \sum_{v=0}^k A_{k-v}^{\alpha} (S_v(x; f_x) - f_x(x)) \right|^{1/(1+\alpha)} \right\}^{1+\alpha} = \\
 & = C_{11} \left(\alpha, \frac{1}{1+\alpha} \right) \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \left| \frac{1}{A_k^{\alpha+1}} \sum_{v=0}^k A_{k-v}^{\alpha} \frac{M}{4(8\pi+1)} \Omega \left(\frac{\pi}{v+1} \right) \right|^{1/(1+\alpha)} \right\}^{1+\alpha} \cong \\
 & \cong \frac{M}{4(8\pi+1)} C_{11} \left(\alpha, \frac{1}{1+\alpha} \right) \left\{ \frac{1}{\log(n+1)} \sum_{k=1}^n \frac{1}{k} \Omega^{1/(1+\alpha)} \left(\frac{1}{k+1} \right) \right\}^{1+\alpha}.
 \end{aligned}$$

Hence, the desired inequality follows with the constant

$$C_4(\alpha) = C_2 C_{11}(\alpha, 1/(1+\alpha))$$

as in the proof of Theorem 2.

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